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A Note on the Hypothesis that $L(s, \chi) > 0$ for all Real Non-Principal Characters χ and for all $s > 0$

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If $\chi(n)$ is an odd real character (mod k), it is an unsolved problem whether

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} > 0 \qquad (s > 0).$$

We propose a hypothesis which ensures the truth of this inequality.

1. As is well known the functions $L(s, \chi)$ for real (non-principal) primitive characters χ, obey a simple functional equation. From this and the extended Riemann hypothesis it immediately follows that:

$$L(s, \chi) \geqslant 0 \ (s > 0) \tag{1}$$

for all such characters χ. Now the restriction that χ is primitive in (1) can be dropped, and we arrive at the conclusion that

$$L(s, \chi) \geqslant 0 \ (s > 0) \tag{2}$$

for all real non-principal characters χ, assuming however that the "e.R.H." is true. The next section contains historical remarks on the conjecture

$$L(s, \chi) > 0 \ (s > 0). \tag{3}$$

Note that this is stronger than (2).

2. Let

$$S_1(x) = \sum_{m \leqslant x} \chi(m),$$

$$S_2(x) = \sum_{m \leqslant x} S_1(m),$$

$$S_3(x) = \sum_{m \leqslant x} S_2(m),$$

and so on. In 1934 S. Chowla (*Acta Arithmetica Band* **1**, 1936) observed that (3) can be proved for many special values of[1] $k \leqslant 200$, when one can prove the existence of an m such that

$$S_m(x) \geqslant 0 \text{ for all } x. \tag{4}$$

In a subsequent paper in the same journal Heilbronn observed that there exist k and real characters χ such that (4) does not hold for any m. Also S. Chowla and A. Selberg[2] showed that

$$L(s, \chi) > 0 \ (0 < s < 1),$$

when χ is the real non-principal character (mod 163). This case had seemed intractable by the method of S. Chowla mentioned above.

3. In connection with the class-number 1 problem recently solved by Stark and Baker, S. Chowla and M. J. DeLeon recently made the following hypothesis (see pp. 261–263 in this issue):

Let k be a square-free number $= 3 \pmod 4$, and let

$$\chi(n) = \left(\frac{n}{k}\right)$$

be the extended Legendre-Jacobi symbol. Let $q_1, \ldots, q_g$ be any sub-set T of the primes $\leqslant k$ such that

$$\left(\frac{q_m}{k}\right) = -1 \qquad (1 \leqslant m \leqslant t).$$

Then there exists a sub-set T such that

$$\sum_{n=1}^{w} \chi^*(n) \geqslant 0 \qquad \text{for all} \quad w \geqslant 1.$$

[1] k is the modulus of the character.

[2] See their paper "On Epstein's zeta Function (I)" in Crelle's journal, 1965.

Here

$$\chi^*(n) = \chi(n)\,\chi_0(n)$$

and $\chi_0(n)$ is the principal character (mod Q) where

$$Q = q_1 ,..., q_g\,.$$

4. In this paper we submit a somewhat weaker hypothesis. We relax the condition that the primes $q_1 ,..., q_t$ should be the primes "$\leqslant k$" with

$$\left(\frac{q_m}{k}\right) = -1 \qquad (1 \leqslant m \leqslant t).$$

We call the new hypothesis J. Thus J is the hypothesis that there exists a set of primes $q_1 ,..., q_g$ with

$$\left(\frac{q_m}{k}\right) = -1 \qquad (1 \leqslant m \leqslant g)$$

and such that

$$(J) \qquad \sum_{j=1}^{w} \chi^*(j) \geqslant 0 \qquad \text{for all} \quad w.$$

Here

$$\chi^*(j) = \chi(j)\,\chi_0(j),$$

where χ_0 is the principal character mod Q and $Q = q_1 q_2 \cdots q_g$, i.e.,

$$\chi_0(j) = \left(\frac{j}{q_1}\right)^2 \left(\frac{j}{q_2}\right)^2 \cdots \left(\frac{j}{q_g}\right)^2.$$

Thus

$$L(s, \chi^*) = L(s, \chi) \prod_{m=1}^{g} \left(1 + \frac{1}{{q_m}^s}\right).$$

Now

$$\begin{aligned} L(s, \chi^*) = {} & \chi^*(1)\left\{\frac{1}{1^s} - \frac{1}{2^s}\right\} \\ & + (\chi^*(1) + \chi^*(2))\left\{\frac{1}{2^s} - \frac{1}{3^s}\right\} \\ & + (\chi^*(1) + \chi^*(2) + \chi^*(3))\left\{\frac{1}{3^s} - \frac{1}{4^s}\right\} \\ & + \cdots \text{ to } \infty, \end{aligned}$$

by the Abel trick. Here the general term is

$$\left\{\sum_{j=1}^{J} \chi^*(j)\right\}\left\{\frac{1}{j^s} - \frac{1}{(j+1)^s}\right\}$$

and so > 0 for all $s > 0$, in virtue of our hypothesis J. Thus

THEOREM. *$L(s, \chi) > 0$ if hypothesis J is true.*

5. We shall call H the hypothesis that

$$L(1, \chi) = \sum_{1}^{\infty} \frac{\chi(n)}{n} > \frac{c}{\log_e k} \, (k > 1),$$

where c is a computable constant independent of k. Hecke proved (see a paper by Landau in Göttinger Nachrichten 1918, *Math. Phys. Klasse*) that the hypothesis L that

$$(L) \qquad L(s, \chi) > 0 \ (s > 0)$$

for real non-principal χ implies H, i.e., we have

$$L \Rightarrow H. \tag{5}$$

In this paper we saw that

$$J \Rightarrow L. \tag{6}$$

Thus

THEOREM. $J \Rightarrow H$.

This is similar to the result proved by Chowla and DeLeon.

6. In this section we note the results of a computation in the case $k = 43$, to prove that J holds in this case and so

$$\sum_{1}^{\infty} \frac{\chi(n)}{n^s} > 0, \qquad (s > 0)$$

when $\chi(n)$ is the real non-principal character (mod 43), i.e., $\chi(n) = (n/43)$. In fact we take $q_1 = 2$, $q_2 = 3$, $q_3 = 5$ when $k = 43$. So $Q = 2 \cdot 3 \cdot 5 = 30$. Now $\chi^*(n)$ is a character (mod 1290) since $43Q = 30 \cdot 43 = 1290$.

The values of $S^*(w) = \sum_1^w \chi^*(n)$ are for $w \leqslant 645$ at intervals given by

$$S^*(w) = 0, 3, 3, 2, 4, 6, 7, 4, 6, 5, 8, 8, 9, 8, 7, 6, 8, 9, 10, 10, 7, 7, 10, 7, 6, 5, 9, 9, 10$$

for

$$w = 7, 17, 27, 37, 47, 57, 67, 77, 87, 97, 107, 117, 127, 137, 147, 157, 167, 177, 187, 197, 207, 217, 277, 307, 347, 457, 487, 547, 617, 643.$$

Throughout $S^*(w) \geqslant 0$ for all $w \geqslant 1$, confirming J at least in this case.